# MATH 33A Worksheet Week 3 Solutions

TA: Emil Geisler and Caleb Partin

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**Exercise 1.** Compute the following or state that it is not defined.

(a) $\begin{bmatrix} 4 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}$

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(a) $\begin{bmatrix} 8 \\ 2 \\ 7 \end{bmatrix}$

(b) $\begin{bmatrix} 13 & 7 \\ 0 & 0 \\ 3 & 2 \end{bmatrix}$

(c) Not defined, $3 \times 3$ and $2 \times 3$ can't be multiplied since $2 \neq 3$. Notice that the other way, $(2 \times 3) \cdot (3 \times 3)$ works.

(d) The $1 \times 1$ matrix $\begin{bmatrix} 5 \end{bmatrix}$. Sometimes we treat this as just a single number, 5.

**Exercise 2.** For each of the following linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$, find the corresponding matrix that represents $T$:

(a) Rotate any vector $\vec{v}$ counter-clockwise by an angle of $\frac{\pi}{2}$ radians

(b) Projection onto the $x$-axis

(c) Projection onto the $y$-axis

(d) First reflect a vector across the line $y = x$, then rotate it by $\frac{\pi}{2}$ radians. (We have matrices $A$ and $B$ that represent both steps of this linear transformation, and a single matrix $C$ that represents the whole transformation. What is the relationship between $A, B$ and $C$?)

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(a) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ Notice that composition of linear transformations is given by multiplication of the corresponding matrices. And order is important, the first transformation we do is given by the rightmost matrix, with subsequent transformations given by multiplication by a matrix on the left.

**Exercise 3.** Let $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots 0 \\ 1 \end{bmatrix}$ be the standard basis vectors of $\mathbb{R}^n$. Show that if $A$ is an $m \times n$ matrix such that $A \cdot e_1 = A \cdot e_2 = \cdots = A \cdot e_n = \vec{0}$, then $A$ is the zero matrix.

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Let $A$ have column vectors $v_1, v_2, \dots, v_m$. Then $A \cdot e_1 = v_1 = \vec{0}$. Moreover, $A \cdot e_i = v_i = 0$ for all the standard basis vectors. Therefore, the first column of $A$ is zero, the second column of $A$ is zero, and so on, so $A$ is all zeroes.

**Exercise 4.** Compute the following for all $\theta \in \mathbb{R}$:

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

What linear transformation do each of these matrices represent? What is the geometric interpretation of the matrix you get as their product?

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$$= \begin{bmatrix} \cos^2(\theta) + \sin^2(\theta) & 0 \\ 0 & \cos^2(\theta) + \sin^2(\theta) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Notice that the first matrix is rotation by $\theta$, and the second matrix is rotation by $-\theta$ since $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$. That is why their product is the identity matrix, since they are inverses as functions $\mathbb{R}^2 \to \mathbb{R}^2$. We'll understand this more when we do inverses.

**Exercise 5. (Challenge Problem):** Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a function which satisfies $\mathbb{R}$-linearity: $F(\vec{v} + a\vec{w}) = F(\vec{v}) + aF(\vec{w})$ for all $\vec{v}, \vec{w} \in \mathbb{R}^n$, $a \in \mathbb{R}$.
Prove that as functions $\mathbb{R}^n \to \mathbb{R}^m$, $F = A$ where $A$ is the matrix with $i$th column vector equal to $F(e_i)$. (Notice that every $\mathbb{R}$-linear function $F : \mathbb{R}^n \to \mathbb{R}^m$ is also *linear*, by letting $\lambda = 1$.) This shows that every $\mathbb{R}$-linear function is a matrix.

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Let $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ by any vector in $\mathbb{R}^n$. Note that by the definition of vector addition in $\mathbb{R}^n$, $\vec{v} = \sum_{i=1}^n v_i e_i$.
First let's see what $A$ does to this vector under matrix multiplication. Recall that the columns of $A$ are given by $F(e_i)$, and so we will get that $A\vec{v} = \sum_{i=1}^n F(e_i) v_i$.
Now let's evaluate $F(\vec{v})$:

$$\begin{aligned} F(\vec{v}) &= F(\sum_{i=1}^n v_i e_i) \\ &= \sum_{i=1}^n F(v_i e_i) \ (\star) \\ &= \sum_{i=1}^n v_i F(e_i) \ (\star) \\ &= A\vec{v}. \end{aligned}$$

The two steps denoted by $(\star)$ are done using the fact that $F$ is given to be $\mathbb{R}$-linear, and so we can commute $F$ with summation and scalar multiplication by scalars in $\mathbb{R}$.

Notice that by these equalities, we can see that $F(\vec{v}) = A\vec{v}$ for all vectors in $\mathbb{R}^n$, and thus they define the same function $\mathbb{R}^n \to \mathbb{R}^n$